

A Note on the Malliavin Differentiability of One-Dimensional Reflected Stochastic Differential Equations with Discontinuous Drift

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Abstract

We consider a one-dimensional Stochastic Differential Equation with reflection where we allow the drift to be merely bounded and measurable. It is already known that such equations have a unique strong solution, see [5]. In [3] and [4] it is shown that non-reflected SDE's with discontinuous drift possess more regularity than one could expect, namely they are Malliavin differentiable and weakly differentiable w.r.t. the initial value. See also [2] for a different technique.

In this paper we use the approach of [3] and [4] to show that similar results hold for one-dimensional SDE's with reflection. We then apply the results to get a Bismut-Elworthy-Li formula for the corresponding Kolmogorov equation.

1 Introduction and Main Result

Let (Ω, \mathcal{F}, P) be a complete probability space with a filtration $\{\mathcal{F}_t\}$ satisfying the usual conditions. Let B_t be a standard \mathcal{F}_t -Brownian motion on the space.

We consider a stochastic differential equation with reflecting boundary:

$$\begin{cases} dX_t &= b(X_t)dt + \sigma(X_t)dB_t + dL_t \\ X_t &\geq 0 \\ L_t &= 1_{\{0\}}(X_t)dL_t, \end{cases} \quad (1)$$

where $b : \mathbb{R} \rightarrow \mathbb{R}$ is a bounded and measurable function and $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ is a continuously differential bounded function, bounded away from zero. This equation has a unique strong solution as proved in [5], namely there exists a pair (X, L) of processes such that

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- X_t is \mathcal{F}_t -adapted, $X_t \geq 0$ for all t .
- L_t is \mathcal{F}_t -adapted, continuous, non-decreasing and such that

$$L_0 = 0, \quad L_t = \int_0^t 1_{\{0\}}(X_s) dL_s,$$

- $X_t = x + \int_0^t b(X_s) ds + \int_0^t \sigma(X_s) B_s + L_t$, $P - a.s.$

We call $x \geq 0$ the initial value of the equation.

For simplicity, we shall consider the equation defined on $t \in [0, 1]$.

The aim of this paper is to show the following

Theorem 1.1. *Assume b is bounded and measurable, $\sigma \in C_b^1(\mathbb{R})$ and there exists $\delta > 0$ such that $|\sigma(x)| \geq \delta$ for all x .*

Then the strong solution to (1) is Malliavin-differentiable, i.e. for a fixed $t \in [0, 1]$, we have $X_t \in \mathbb{D}^{1,2}$.

The outline of this paper is as follows: The rest of this section is devoted to the proof of Theorem 1.1. In Section 2 we study how the solution depends on the initial value $x \geq 0$. We then use this to study the corresponding Kolmogorov equation and obtain a Bismut-Elworthy-Li formula in Section 3. Section 4 is the Appendix where we include an approximation of the Skorohod equation.

We return to the proof of Theorem 1.1 which is divided into three steps. In the two first steps we consider (1) with a drift $b \in C_b^1(\mathbb{R})$ such that $b(0) = 0$. In *step 1* we introduce an approximation of the solution in terms of an ordinary SDE, i.e. not reflected.

In *step 2* we use the approximation from step 1 to find bounds on the Malliavin derivative which are not depending on b' .

In *step 3* we consider a general b and construct an approximation of the solution such that the sequence of Malliavin derivatives are bounded uniformly.

Step 1

The function $(\cdot)^- : \mathbb{R} \rightarrow \mathbb{R}$ is defined by

$$(y)^- = \begin{cases} -y & \text{if } y < 0 \\ 0 & \text{if } y \geq 0. \end{cases}$$

Let $\rho \in C^\infty(\mathbb{R})$ be a positive function such that $\text{supp}\{\rho\} \subset (-1, 1)$ and $\int \rho(z) dz = 1$. Define $\rho_n(z) = n\rho(nz)$ and let

$$h_n(y) = \int \rho_n(y - z)(z)^- dz.$$

It is readily checked that $h_n \in C^\infty(\mathbb{R})$, $h'_n(z) \leq 0$ and $h_n \rightarrow (\cdot)^-$ almost everywhere. Then there exists a unique strong solution to the SDE

$$X_t^{n,\epsilon} = x + \int_0^t b(X_s^{n,\epsilon}) + \frac{1}{\epsilon} h_n(X_s^{n,\epsilon}) ds + \int_0^t \sigma(X_s^{n,\epsilon}) dB_s.$$

As $n \rightarrow \infty$ it is easy to see that $X_t^{n,\epsilon} \rightarrow X_t^\epsilon$ in $L^2(\Omega)$, where

$$X_t^\epsilon = x + \int_0^t b(X_s^\epsilon) + \frac{1}{\epsilon} (X_s^\epsilon)^- ds + \int_0^t \sigma(X_s^\epsilon) dB_s.$$

The following lemma is a classical result. We include a proof for the sake of self-containdness.

Lemma 1.2. *As $\epsilon \rightarrow 0$, we get $(X_t^\epsilon, \epsilon^{-1} \int_0^t (X_s^\epsilon)^- ds) \rightarrow (X_t, L_t)$ - the solution to (1).*

Proof. By the comparison principle, we note that there exists a subset with full measure $\Omega_0 \subset \Omega$ such that $\{X_t^\epsilon(\omega)\}$ is increasing as $\epsilon \rightarrow 0$ for all $(t, \omega) \in [0, 1] \times \Omega_0$. We may define

$$X_t(\omega) = \lim_{\epsilon \rightarrow 0} X_t^\epsilon(\omega)$$

for $\omega \in \Omega_0$ and 0 otherwise.

By Itô's formula we have

$$\begin{aligned} (X_t^\epsilon)^2 &= x^2 + \int_0^t 2X_s^\epsilon(\epsilon^{-1}(X_s^\epsilon)^- + b(X_s)) + \sigma^2(X_s^\epsilon) ds + \int_0^t 2X_s^\epsilon \sigma(X_s^\epsilon) dB_s \\ &\leq x^2 + \int_0^t 2X_s^\epsilon b(X_s) + \sigma^2(X_s^\epsilon) ds + \int_0^t 2X_s^\epsilon \sigma(X_s^\epsilon) dB_s \end{aligned} \tag{2}$$

and taking expectation yields

$$\begin{aligned} E[(X_t^\epsilon)^2] &\leq x^2 + \int_0^t E[2X_s^\epsilon b(X_s)] + E[\sigma^2(X_s^\epsilon)] ds \\ &\leq x^2 + \|b\|_\infty^2 t + \|\sigma\|_\infty^2 t + \int_0^t E[(X_s^\epsilon)^2] ds \\ &\leq (x^2 + \|b\|_\infty^2 t + \|\sigma\|_\infty^2 t) e^t \end{aligned}$$

where we have used the inequality $2ab \leq a^2 + b^2$ and Gronwall's inequality.

It follows by Fatou's lemma that X_t is P -a.s. finite.

Define Y_t^ϵ to be the solution of

$$dY_t^\epsilon = \epsilon^{-1}(Y_t^\epsilon)^- dt + b(X_t) dt + \sigma(X_t) dB_t, \quad Y_0^\epsilon = x.$$

From Proposition 4.3 we get that on the subset on which $\int_0^\cdot b(X_s)ds + \int_0^\cdot \sigma(X_s)dB_s$ is continuous, we have that Y^ϵ (respectively $\epsilon^{-1} \int_0^\cdot (Y_s^\epsilon)^- ds$) converges to Y (respectively ϕ) in $C([0, 1])$ which is the solution to the Skorohod equation.

We have by Itô's formula,

$$\begin{aligned} (X_t^\epsilon - Y_t^\epsilon)^2 &= \frac{2}{\epsilon} \int_0^t ((X_s^\epsilon)^- - (Y_s^\epsilon)^-) (X_s^\epsilon - Y_s^\epsilon) ds \\ &\quad + 2 \int_0^t (X_s^\epsilon - Y_s^\epsilon) (b(X_s^\epsilon) - b(X_s)) ds \\ &\quad + 2 \int_0^t (X_s^\epsilon - Y_s^\epsilon) (\sigma(X_s^\epsilon) - \sigma(X_s)) dB_s \\ &\quad + \int_0^t (\sigma(X_s^\epsilon) - \sigma(X_s))^2 ds. \end{aligned}$$

The first term above is always negative. Taking expectation we get :

$$\begin{aligned} E[(X_t^\epsilon - Y_t^\epsilon)^2] &\leq 2 \int_0^t E[(X_s^\epsilon - Y_s^\epsilon) (b(X_s^\epsilon) - b(X_s))] ds \\ &\quad + \int_0^t E[(\sigma(X_s^\epsilon) - \sigma(X_s))^2] ds \\ &\leq \int_0^t E[(X_s^\epsilon - Y_s^\epsilon)^2] ds + \int_0^t E[(b(X_s^\epsilon) - b(X_s))^2] ds \\ &\quad + \int_0^t E[(\sigma(X_s^\epsilon) - \sigma(X_s))^2] ds \\ &\leq e^t \left(\int_0^t E[(b(X_s^\epsilon) - b(X_s))^2] ds + \int_0^t E[(\sigma(X_s^\epsilon) - \sigma(X_s))^2] ds \right). \end{aligned}$$

Above we have used Gronwall's lemma in the last inequality. As $\epsilon \rightarrow 0$ the above goes to zero, and we see that $Y_t^\epsilon \rightarrow X_t$ P -a.s. for all $t \in [0, 1]$. Using the Burkholder-Davies-Gundy inequality, one can show that this convergence actually takes place in $L^2(\Omega; C([0, 1]))$.

It follows from Proposition 4.3 that X_t is continuous and that $\epsilon^{-1} \int_0^\cdot (X_s^\epsilon)^- ds$ converges to L where (X, L) is a solution (1).

□

Step 2

We have the following estimate on the Malliavin derivatives:

Lemma 1.3. *For fixed $t \geq 0$ we have $X_t \in \mathbb{D}^{1,2}$ and there exists an increasing function $K_1 : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that the Malliavin derivative satisfies*

$$E[(D_\theta X_t)^2] \leq K_1(\|\sigma\|_{C_b^1}) \left(E[\exp\{4 \int_\theta^t b'(X_s) ds\}] \right)^{1/2}.$$

Proof. We observe that $X_t^{n,\epsilon} \in \mathbb{D}^{1,2}$ and the Malliavin derivative satisfies

$$D_\theta X_t^{n,\epsilon} = \sigma(X_\theta^{n,\epsilon}) + \int_\theta^t (b'(X_s^{n,\epsilon}) + \epsilon^{-1} h'_n(X_s^{n,\epsilon})) D_\theta X_s^{n,\epsilon} ds + \int_\theta^t \sigma'(X_s^{n,\epsilon}) D_\theta X_s^{n,\epsilon} dB_s. \quad (3)$$

This is a linear SDE which is uniquely solved by

$$\begin{aligned} D_\theta X_t^{n,\epsilon} &= \sigma(X_\theta) \exp \left\{ \int_\theta^t b'(X_s^{n,\epsilon}) + \epsilon^{-1} h'_n(X_s^{n,\epsilon}) - \frac{1}{2} (\sigma'(X_s^{n,\epsilon}))^2 ds \right\} \\ &\quad \times \exp \left\{ \int_\theta^t \sigma'(X_s^{n,\epsilon}) dB_s \right\} \\ &\leq \exp \left\{ \int_\theta^t b'(X_s^{n,\epsilon}) ds \right\} \exp \left\{ \int_\theta^t \sigma'(X_s^{n,\epsilon}) dB_s \right\} \end{aligned}$$

since h'_n is negative.

Using that for a bounded adapted process $\{\psi(s)\}_{s \in [0,1]}$ we have

$$E[\exp\{\int_\theta^t \psi(s) dB_s\}] \leq \exp \left\{ \frac{(t-\theta)}{2} \|\psi\|_\infty^2 \right\}$$

and Hölder's inequality, we get

$$E[(D_\theta X_t^{n,\epsilon})^2] \leq \|\sigma\|_\infty^2 \exp\{c\|\sigma'\|_\infty^2\} \left(E[\exp\{4 \int_\theta^t b'(X_s^{n,\epsilon}) ds\}] \right)^{1/2}.$$

Letting first n go to infinity and ϵ tend to zero we get the result. \square

What is left is to find a bound on $E[\exp\{4 \int_\theta^t b'(X_s^{n,\epsilon}) ds\}]$ that is depending only on $\|b\|_\infty$.

The following Proposition is based on Proposition 3 in [1].

Proposition 1.4. *There exists a constant C such that for every positive integer k we have*

$$E \left(\int_\theta^t b'(X_s) ds \right)^k \leq \frac{C^k \|b\|_\infty^k |t-\theta|^{k/2} k!}{\Gamma(\frac{k}{2} + 1)}$$

for all $b \in C_c^\infty((0, \infty))$.

Let us explain briefly the idea of the proof. Using the Markov property we can write the above left-hand-side as

$$\int_{\theta < t_1 < \dots < t_k < t} \int_{\mathbb{R}_+^k} \prod_{j=1}^k b'(z_j) P(t_j - t_{j-1}, z_j, z_{j-1}) dz_k \dots dz_1 dt_1 \dots dt_k,$$

where P is the transition density of (1). Then use integration by parts to move the derivatives onto the density function. Then one can show the result by using estimates on P and its derivatives.

Let us remark that the proof of Proposition 1.4 is the same as the proof of Proposition 3 in [1] when we replace Lemma 1 in [1] by the following:

Lemma 1.5. *The operator $T : L^2([0, 1] \times \mathbb{R}_+) \rightarrow L^2([0, 1] \times \mathbb{R}_+)$ defined by*

$$Th(s, y) = \int_s^1 \int_{\mathbb{R}_+} \partial_x \partial_y P(t - s, y, z) h(t, z) dz dt$$

is bounded.

We note that $P(t, x, y)$ the fundamental solution to

$$\partial_t u = b \partial_x u + \frac{1}{2} \sigma^2 \partial_x^2 u, \quad \partial_x u|_{t=0} = 0.$$

Lemma 1.5 follows from a 'T(1) theorem on spaces of homogeneous type' using the Schauder estimates obtained in the following lemma:

Lemma 1.6. *We equip $[0, 1] \times \mathbb{R}$ with the parabolic metric $d(t, x) = \sqrt{t} + |x|$. There exists constants $C, c > 0$ such that we have*

- $|P(t, x, y)| \leq C t^{-1/2} \exp\{-\frac{c(x-y)^2}{t}\}$
- $|\partial_x \partial_y P(t, x, y)| \leq C t^{-3/2} \exp\{-\frac{c(x-y)^2}{t}\}$
- $|\partial_x \partial_y P(t, x, y)| \leq d(t, x - y)^{-3}$
- $|\partial_x \partial_y P(t - s, x, y) - \partial_x \partial_y P(t' - s, x, y')| \leq C \frac{d(t-t', y-y')^\delta}{d(t-s, x-y)^{3+\delta}}$ for some $\delta > 0$, whenever $\frac{d(t-t', y-y')}{d(t-s, x-y)} < \frac{1}{2}$.
- $\int_{\mathbb{R}} \partial_x \partial_y P(t, x, y) dy = \int_{\mathbb{R}} \partial_x \partial_y P(t, x, y) dx = 0$

Combining Proposition 1.4 and Lemma 1.3 we are able finish step 1:

Proposition 1.7. *There exists a continuous function $C_\delta : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$ increasing in both variables such that*

$$E[(D_\theta X_t)^2] \leq C_\delta(\|b\|_\infty, \|\sigma\|_{C_b^2}).$$

Moreover, C_δ is independent of t and θ .

Remark 1.8. By approximation, one can get the same estimate as in Proposition 1.7 when assuming that b is Lipschitz continuous.

We now turn to step 3 of our proof, which concludes the proof of Theorem 1.1.

Step 3

Proof of Theorem 1.1. Assume $b : \mathbb{R} \rightarrow \mathbb{R}$ is bounded and measurable. Choose a function $\psi \in C^\infty$ such that

$$\psi(y) = \begin{cases} 1 & \text{if } y \geq 1 \\ 0 & \text{if } y \leq 0 \end{cases}.$$

For $n \in \mathbb{N}$ we define $\psi_n^0(y) = \psi(ny)$, $\psi_n^1(y) = 1 - \psi(n^{-1}y - n)$ and $\psi_n(y) = \psi_n^0(y) + \psi_n^1(y)$. It is readily checked that ψ_n is smooth and has compact support. Moreover, $\psi_n(0) = 0$ and

$$\lim_{n \rightarrow \infty} \psi_n(y) = \begin{cases} 1 & \text{if } y > 0 \\ 0 & \text{if } y \leq 0 \end{cases}.$$

Define $b_j(y) := \int \rho_j(y - z)b(z)dz\psi_n(y)$, and let

$$b_{n,k} := \bigwedge_{j=1}^k b_j,$$

and

$$\hat{b}_n = \bigwedge_{j=n}^{\infty} b_j.$$

Then $b_{n,k}$ is Lipschitz continuous, $b_{n,k}(0) = 0$, \hat{b}_n , $b_{n,k}$ are uniformly bounded and we have

$$b_{n,k} \geq b_{n,k+1} \geq \cdots \rightarrow \hat{b}_n, \quad \text{as } k \rightarrow \infty,$$

and

$$\hat{b}_n \leq \hat{b}_{n+1} \rightarrow b \quad \text{as } n \rightarrow \infty$$

almost surely with respect to Lebesgue measure.

Using the comparison theorem for SDE's one can show that for the corresponding sequences of solutions, denoted $(X^{n,k}, L^{n,k})$ and (X^n, L^n) , we have the following convergence in $L^2(\Omega)$:

$$(X_t^n, L_t^n) = \lim_{k \rightarrow \infty} (X_t^{n,k}, L_t^{n,k}) \quad \text{uniformly in } t$$

and

$$(X_t, L_t) = \lim_{n \rightarrow \infty} (X_t^n, L_t^n) \quad \text{uniformly in } t$$

where (X, L) is a solution to (1). Details can be found in [5].

By Proposition 1.7 we have $\sup_{n,k \geq 1} \|X_t^{n,k}\|_{1,2} < \infty$. The result follows. \square

2 Spatial Regularity

In this section we want to emphasise that the equation (1) depends on the initial value $x \geq 0$. We write $X_t(x)$ for the unique strong solution.

Proposition 2.1. *The solution to (1) is locally weakly differentiable in the sense that for a bounded, open subset $U \subset \mathbb{R}_+$ and any $p > 1$ we have*

$$X_t(\cdot) \in L^2(\Omega; W^{1,p}(U)).$$

The proof follows the same steps as in the previous section and we just indicate the proof here.

For the first step we assume $b \in C_b^1(\mathbb{R})$, $b(0) = 0$, and we consider the approximating sequence of solutions

$$X_t^{n,\epsilon}(x) = x + \int_0^t b(X_s^{n,\epsilon}(x)) + \frac{1}{\epsilon} h_n(X_s^{n,\epsilon}(x)) ds + \int_0^t \sigma(X_s^{n,\epsilon}(x)) dB_s.$$

Then the solution is in C^1 and we have that the spatial derivative satisfies

$$\begin{aligned} \partial_x X_t^{n,\epsilon}(x) &= 1 + \int_0^t b'(X_s^{n,\epsilon}(x)) \partial_x X_s^{n,\epsilon}(x) ds \\ &\quad + \int_0^t \frac{1}{\epsilon} h'_n(X_s^{n,\epsilon}(x)) \partial_x X_s^{n,\epsilon}(x) ds + \int_0^t \sigma'(X_s^{n,\epsilon}(x)) \partial_x X_s^{n,\epsilon}(x) dB_s. \end{aligned}$$

We recognize this equation as the same as (3) when we let $\theta = 0$. It is then easy to see that the results of Lemma 1.3, Propositions 1.4 and 1.7 when we replace the Malliavin derivative by the spatial derivative.

More specifically, we in place of Proposition 1.7 we get that when b is Lipschitz,

$$\sup_{x \geq 0} E |\partial_x X_t(x)|^p \leq C(\|\sigma\|_{C_b^1}, \|b\|_\infty).$$

Since $U \subset \mathbb{R}_+$ is bounded we see that $X_t(\cdot) \in L^2(\Omega; W^{1,p}(U))$.

If now b is merely bounded and measurable we use the same method as step 2 in the previous section to conclude:

Lemma 2.2. *There exists a sequence $X_t^k(\cdot)$, bounded in $L^2(\Omega; W^{1,p}(U))$, such that $X_t^k(x) \rightarrow X_t(x)$ in $L^2(\Omega)$ for all $x \geq 0$.*

We arrive at the proof of Proposition 2.1

Proof of 2.1. From Lemma 2.2 we get that there exists a subsequence $\{X_t^{k_j}(\cdot)\}_{j \geq 1}$ that is converging in the weak topology of $L^2(\Omega; W^{1,p}(U))$ to

some element Y_t . Since $X_t^{k_j}(x) \rightarrow X_t(x)$, we have for any $A \in \mathcal{F}$ and $\varphi \in C_c^\infty(U)$

$$\begin{aligned} E[1_A \int_U \varphi'(x) X_t(x) dx] &= \lim_{j \rightarrow \infty} E[1_A \int_U \varphi'(x) X_t^{k_j}(x) dx] \\ &= - \lim_{j \rightarrow \infty} E[1_A \int_U \varphi(x) \partial_x X_t^{k_j}(x) dx] \\ &= -E[1_A \int_U \varphi(x) \partial_x Y_t(x) dx]. \end{aligned}$$

It follows that $X_t(\cdot)$ is P -a.s. weakly differentiable and its weak derivative is equal to $\partial_x Y_t(x)$. □

3 Bismut-Elworthy-Li Formula

In this section we study the PDE

$$\partial_t u(t, x) = b(x) \partial_x u(t, x) + \frac{1}{2} \sigma^2(x) \partial_x^2 u(t, x), \text{ for } x \geq 0 \quad (4)$$

with initial and boundary condition

$$u(0, x) = u_0(x), \quad \partial_x u(t, 0) = 0.$$

We shall use the same assumptions on b and σ as in Theorem 1.1 and $u_0 \in C_b^1(\mathbb{R}_+)$.

Existence and uniqueness of a solution to (4) is already known. More specifically, the solution is given by

$$u(t, x) = E[u_0(X_t(x))]$$

and lies in $W_{loc}^{(1,2),p}((0, 1] \times \mathbb{R}_+)$ - the space of functions which are once weakly differentiable w.r.t t and twice weakly differentiable w.r.t. x and these functions are locally p -integrable. Moreover, the solution is in $C([0, 1] \times \mathbb{R}_+)$.

In this section, however, we shall prove a Bismut-Elworthy-Li formula for the derivative of the solution to (4) which does not depend on the derivative of u_0 .

Theorem 3.1. *For a bounded subset $U \subset \mathbb{R}_+$ the (weak) spatial derivative of u takes the form*

$$\partial_x u(t, x) = E[u_0(X_t(x)) t^{-1} \int_0^t \partial_x X_s(x) dB_s], \quad (5)$$

for almost every $x \in U$.

Proof. As in the proof of Proposition 2.1 we have a sequence of processes $\{X_t^k(x)\}$ that are P -a.s. differentiable in x , $X_t^k(x) \rightarrow X_t(x)$ in $L^2(\Omega)$ and $X_t^k(\cdot)$ converges to $X_t(\cdot)$ in the weak topology of $L^2(\Omega; W^{1,p}(U))$.

We certainly get that

$$u_k(t, x) := E[u_0(X_t^k(x))] \rightarrow u(t, x)$$

as $k \rightarrow \infty$ for every $(t, x) \in [0, 1] \times \mathbb{R}_+$. We will now show that $\partial_x u_k(t, \cdot)$ converges weakly to the right-hand-side of (5), thus proving the assertion.

We start by noting that $\partial_x X_t^k(x) = D_s X_t^k(x) \partial_x X_s^k(x)$, and by the chain-rule for the Malliavin derivative we have

$$\begin{aligned} u'_0(X_t^k(x)) \partial_x X_t^k(x) &= u'_0(X_t^k(x)) t^{-1} \int_0^t D_s X_t^k(x) \partial_x X_s^k(x) ds \\ &= t^{-1} \int_0^t D_s(u_0(X_t^k(x))) \partial_x X_s^k(x) ds. \end{aligned}$$

Taking expecations in the above formula and using the duality between the Malliavin derivative and the Itô-integral we get

$$\begin{aligned} \partial_x u_k(t, x) &= E[u'_0(X_t^k(x)) \partial_x X_t^k(x)] \\ &= t^{-1} E\left[\int_0^t D_s(u_0(X_t^k(x))) \partial_x X_s^k(x) ds\right] \\ &= t^{-1} E\left[u_0(X_t^k(x)) \int_0^t \partial_x X_s^k(x) dB_s\right]. \end{aligned}$$

For a test function $\varphi \in C_c^\infty(U)$ we have

$$\begin{aligned} \int_U \varphi'(x) u(t, x) dx &= - \lim_{k \rightarrow \infty} \int_U \varphi(x) \partial_x u_k(t, x) dx \\ &= - \lim_{k \rightarrow \infty} \int_U \varphi(x) t^{-1} E[u_0(X_t^k(x)) \int_0^t \partial_x X_s^k(x) dB_s] dx. \\ &= - \lim_{k \rightarrow \infty} \int_U \varphi(x) t^{-1} E[(u_0(X_t^k(x)) - u_0(X_t(x))) \int_0^t \partial_x X_s^k(x) dB_s] dx. \\ &\quad - \lim_{k \rightarrow \infty} \int_U \varphi(x) t^{-1} E[u_0(X_t(x)) \int_0^t \partial_x X_s^k(x) dB_s] dx. \end{aligned}$$

To see that the first term converges to zero, note that for all $x \in U$

$$\begin{aligned}
& E[(u_0(X_t^k(x)) - u_0(X_t(x))) \int_0^t \partial_x X_s^k(x) dB_s] \\
& \leq \|u'\|_\infty \|X_t^k(x) - X_t(x)\|_{L^2(\Omega)} \left\| \int_0^t \partial_x X_s^k(x) dB_s \right\|_{L^2(\Omega)} \\
& = \|u'\|_\infty \|X_t^k(x) - X_t(x)\|_{L^2(\Omega)} \left(\int_0^t E[|\partial_x X_s^k(x)|^2] ds \right)^{1/2} \\
& \leq \|u'\|_\infty \|X_t^k(x) - X_t(x)\|_{L^2(\Omega)} \left(t \sup_{m,y,s} E[|\partial_x X_s^m(y)|^2] \right)^{1/2}
\end{aligned}$$

which converges to zero as $k \rightarrow \infty$.

For the second term, notice that since $X_t(x)$ is Malliavin differentiable and $u_0 \in C_b^1(\mathbb{R}_+)$, we have by the Clark-Ocone formula

$$u_0(X_t(x)) = E[u_0(X_t(x))] + \int_0^t E[D_s u_0(X_t(x)) | \mathcal{F}_s] dB_s$$

and so

$$\begin{aligned}
& \lim_{k \rightarrow \infty} \int_U \varphi(x) t^{-1} E[u_0(X_t(x)) \int_0^t \partial_x X_s^k(x) dB_s] dx \\
& = \lim_{k \rightarrow \infty} \int_U \varphi(x) t^{-1} E\left[\int_0^t D_s(u_0(X_t(x))) \partial_x X_s^k(x) ds\right] dx \\
& = \int_U \varphi(x) t^{-1} E\left[\int_0^t D_s(u_0(X_t(x))) \partial_x X_s(x) ds\right] dx
\end{aligned}$$

where we have used dominated convergence w.r.t s and weak convergence w.r.t. x .

We finally note that

$$E\left[\int_0^t D_s(u_0(X_t(x))) \partial_x X_s(x) ds\right] = E[u_0(X_t(x)) \int_0^t \partial_x X_s(s) dB_s]$$

again by the Clark-Ocone formula.

□

4 Appendix: The Skorohod equation

Given a continuous function g such that $g(0) = 0$ and $x \geq 0$ we are searching for nondecreasing function $\phi \in C([0, 1])$ such that if we define

$$f(t) := x + \phi(t) + g(t),$$

then $f(t) \geq 0$ for all $t \in C([0, 1])$ and $\int_0^1 1_{\{f(s) > 0\}} d\phi(s) = 0$. We call the pair (f, ϕ) a solution to the Skorohod equation if they satisfies the above.

It is well known that such a solution exists and it is uniquely given by

$$\phi(t) = \max\{0, \max_{0 \leq s \leq t} -(x + g(s))\}.$$

The topic of this Appendix is however to approximate the solution in a suitable sense.

Let $\epsilon > 0$ and denote by f^ϵ the solution of the following ODE:

$$f^\epsilon(t) = x + \frac{1}{\epsilon} \int_0^t (f^\epsilon(s))^- ds + g(t). \quad (6)$$

We have:

Lemma 4.1. *Assume $g \in C^1([0, 1])$, $g(0) = 0$. As $\epsilon \rightarrow 0$, there exists a subsequence of $(f^\epsilon, \epsilon^{-1} \int_0^\cdot (f^\epsilon(s))^- ds)$ converging uniformly to (f, ϕ) - the solution to the Skorohod equation.*

Proof. By the comparison principle for ODE's we have that f^ϵ is pointwise increasing to some function denoted f . We have

$$\begin{aligned} (f^\epsilon(t))^2 &= x^2 + \frac{2}{\epsilon} \int_0^t f^\epsilon(s) (f^\epsilon(s))^- ds + 2 \int_0^t f^\epsilon(s) \dot{g}(s) ds \\ &\leq x^2 + \int_0^t (f^\epsilon(s))^2 + (\dot{g}(s))^2 ds \\ &\leq \left(x^2 + \int_0^t (\dot{g}(s))^2 ds \right) e^t \end{aligned}$$

where the last inequality comes from Gronwalls lemma. It follows that $f(t) < \infty$ for all t .

Furthermore, we have

$$\begin{aligned} (f^\epsilon(t))^- &= \frac{1}{\epsilon} \int_0^t 1_{(f^\epsilon \leq 0)} (f^\epsilon(s))^- ds + \int_0^t 1_{(f^\epsilon \leq 0)} \dot{g}(s) ds \\ &= \frac{1}{\epsilon} \int_0^t (f^\epsilon(s))^- ds + \int_0^t 1_{(f^\epsilon \leq 0)} \dot{g}(s) ds \\ &= \int_0^t 1_{(f^\epsilon \leq 0)} \dot{g}(s) e^{-\epsilon^{-1}(t-s)} ds \end{aligned}$$

where the last inequality comes from solving the linear ODE that $(f^\epsilon(t))^-$ satisfies. It follows from the above that

$$(f^\epsilon(t))^- \leq \|\dot{g}\|_\infty \epsilon.$$

From (6) and the above shows that

$$\dot{f}^\epsilon(t) = \epsilon^{-1}f^\epsilon(t) + \dot{g}(s)$$

is then uniformly bounded by $2\|\dot{g}\|_\infty$. Using the relative compactness of \dot{f}^ϵ in $L^2([0,1])$ with respect to the weak topology we can extract a converging subsequence (still denoted \dot{f}^ϵ for simplicity). Denote the limit by \tilde{f} .

Then,

$$f(t) = \lim_{\epsilon \rightarrow 0} f^\epsilon(t) = x + \lim_{\epsilon \rightarrow 0} \int_0^1 1_{[0,t]}(s) \dot{f}^\epsilon(s) ds = x + \int_0^1 1_{[0,t]}(s) \tilde{f}(s) ds$$

so that f is continuous. It follows from Dini's theorem that the convergence is uniform in t .

To see that $f(t)$ is positive assume that there exists t_0 such that $f(t_0) < 0$. By continuity we may choose $\delta > 0$ such that $f(t) \leq \frac{f(t_0)}{2}$ for all $t \in (t_0 - \delta, t_0 + \delta)$. Moreover, by the uniform convergence there exists $\epsilon_0 > 0$ such that

$$f^\epsilon(t) \leq \frac{f(t_0)}{4}, \forall t \in (t_0 - \delta, t_0 + \delta) \text{ and } \forall \epsilon < \epsilon_0.$$

It follows that

$$\begin{aligned} f^\epsilon(t_0 + \frac{\delta}{2}) - f^\epsilon(t_0 - \frac{\delta}{2}) &= \int_{t_0 - \delta/2}^{t_0 + \delta/2} (f^\epsilon(s))^- ds + g(t_0 + \frac{\delta}{2}) - g(t_0 - \frac{\delta}{2}) \\ &\geq -\frac{f(t_0)\delta}{4\epsilon} + g(t_0 + \frac{\delta}{2}) - g(t_0 - \frac{\delta}{2}) \\ &\rightarrow +\infty \end{aligned}$$

as $\epsilon \rightarrow 0$ which contradicts the finiteness of f . Consequently, $f(t) \geq 0$ for all t .

It is clear from (6) that also $\epsilon^{-1} \int_0^1 (f^\epsilon(s))^- ds$ is converging in $C([0,1])$, and we denote the limit by $\phi(t)$. Being the limit of a sequence of nondecreasing functions, ϕ itself is increasing. Moreover, we have that ϕ is constant on $\{t \in [0,1] | f(t) > 0\}$. Indeed, assume $f(t) > \gamma > 0$ for all $t \in (a,b) \subset [0,1]$. We may choose $\epsilon_0 > 0$ such that $f^\epsilon(t) \geq \gamma/2 > 0$ for all $t \in (a,b)$.

For $r < s \in (a,b)$ we have

$$\phi(s) - \phi(r) = \lim_{\epsilon \rightarrow 0} \epsilon^{-1} \int_r^s (f^\epsilon(u))^- du = 0,$$

the claim follows and we get $\int_0^1 1_{(0,\infty)}(f(s)) d\phi(s) = 0$.

□

We write f_g to emphasize that the above function depends on g . We can then get the following

Lemma 4.2. *The mapping $g \mapsto f_g$ can be extended to a Lipschitz-continuous mapping from $C([0, 1])$ into itself.*

Proof. If we denote by f_j^ϵ the solution to (6) when we replace g by $g_j \in C^1([0, 1])$, $j = 1, 2$, it is enough to find the uniform bound

$$\|f_1^\epsilon - f_2^\epsilon\|_\infty \leq 2\|g_1 - g_2\|_\infty. \quad (7)$$

To this end, define the functions

$$h_j^\epsilon(t) := f_j^\epsilon(t) - g_j(t) = x + \int_0^t (h_j^\epsilon(s) + g_j(s))^- ds.$$

With $K := \|g_1 - g_2\|_\infty$ we have

$$\begin{aligned} & (h_1^\epsilon(t) - h_2^\epsilon(t) - K)^+ \\ &= \int_0^t 1_{(h_1^\epsilon(s) - h_2^\epsilon(s) > K)} ((h_1^\epsilon(s) + g_1(s))^- - (h_2^\epsilon(s) + g_2(s))^-) ds. \end{aligned}$$

We see that the above integrand is negative for all s . Indeed, fix $s \in [0, t]$. If $(h_1^\epsilon(s) + g_1(s))^- - (h_2^\epsilon(s) + g_2(s))^-$ is negative we are done. If $(h_1^\epsilon(s) + g_1(s))^- - (h_2^\epsilon(s) + g_2(s))^-$ is positive, we write

$$h_1^\epsilon(s) - h_2^\epsilon(s) \leq h_1^\epsilon(s) + g_1(s) - h_2^\epsilon(s) + g_2(s) + K$$

so that

$$1_{(h_1^\epsilon(s) - h_2^\epsilon(s) > K)} \leq 1_{(h_1^\epsilon(s) + g_1(s) - h_2^\epsilon(s) + g_2(s) > 0)}.$$

It is then easy to check that the function $(x, y) \mapsto 1_{(x-y)>0}((x)^- - (y)^-)$ is always non-positive. □

We are ready to conclude:

Proposition 4.3. *Let $g \in C([0, 1])$ be such that $g(0) = 0$ and $x \geq 0$. Then the solution to*

$$f_g^\epsilon(t) = x + \epsilon^{-1} \int_0^t (f_g^\epsilon(s))^- ds + g(t)$$

converges in $C([0, 1])$ as $\epsilon \rightarrow 0$ to the solution to the Skorohod equation.

Proof. Let $\delta > 0$. Choose $g_\delta \in C^1([0, 1])$ such that $\|g - g_\delta\|_\infty < \delta$. By Lemma 4.1 we can choose $\epsilon_0 > 0$ such that for all $\epsilon \in (0, \epsilon_0)$ we have $\|f_{g_\delta} - f_{g_\delta}^\epsilon\|_\infty < \delta$. By the proof of Lemma 4.2 we get

$$\begin{aligned} \|f_g - f_g^\epsilon\|_\infty &\leq \|f_g - f_{g_\delta}\|_\infty + \|f_{g_\delta} - f_{g_\delta}^\epsilon\|_\infty + \|f_{g_\delta}^\epsilon - f_g^\epsilon\|_\infty \\ &< 2\|g - g_\delta\| + \delta + 2\|g - g_\delta\|_\infty = 5\delta. \end{aligned}$$

The conditions of the Skorohod equation are easy to check. □

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